Three Types of Chaotic Attractors in 3D Maps

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Coupling a one-dimensional chaotic forcing to a stable fixed point in the plane may generate different fractal attractors embedded in three dimensions. The system with real eigenvalues of the fixed point gives rise to simple chaotic attractors with *three* different types of fractal structures. We show that the competition of local exponents provides a generic criterion for the classification of the fractal structures in dynamical systems.

1. Introduction

It was shown [1, 2] that simple chaotic attractors of three-variable time-discrete maps with a spectrum of ordered Lyapunov characteristic exponents (LCEs) of $(+,-,-)$, (where $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$ and $\sum_i \Lambda_i < 0$, $i = 1, 2, 3$), may possess *three* different types of fractal structures. These are distinguishable by means of their two-by-two sums of LCEs.

The fractal structure of chaotic attractors can be predicted from the relation of the *global* means of divergence (positive exponent) and the means of convergence (negative exponents) [3, 4] classified by their respective sums. The lines of transition between these types of chaotic attractors are given with $\Lambda_1 + \Lambda_2 = 0$ and $\Lambda_1 + \Lambda_3 = 0$, respectively.

2. The Map

Here we investigate a simple three-variable map which realizes the coupling of a chaotic variable to a two-dimensional subsystem with an attracting fixed point. For the sake of direct computability of the Lyapunov exponents Λ_i from the eigenvalues λ_i without averaging of local exponents we take the piecewise linear map

$$\begin{aligned} x_{i+1} &= (C\, x_i) \bmod 1\,, \\ y_{i+1} &= \varepsilon\, x_i + A\, y_i - B\, z_i\,, \\ z_{i+1} &= y_i \end{aligned} \tag{1}$$

with variables $x, y, z \in \mathbb{R}$ and parameters $A, B, C, \varepsilon \in \mathbb{R}$, $i \in \mathbb{N}$. The chaotic forcing in x (piecewise-linear Bernouilli modulo map) has an eigenvalue (local divergence) $\lambda_1 = C$ and LCE $\Lambda_1 = \ln C$, respectively, for $C > 1$ in the intervall [0, 1].

For $\varepsilon = 0$ the linear two-variable subsystem (y, z) has a fixed point at the origin with eigenvalues

$$\lambda_{2,3} = \tfrac{1}{2}\left(A \pm \sqrt{A^2 - 4B}\right). \tag{2}$$

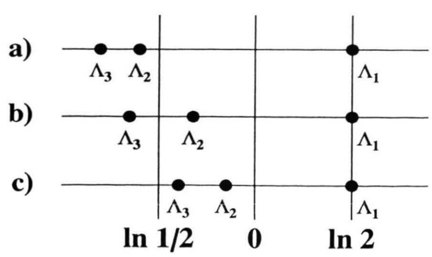

Fig. 1. Possible types of ordinary chaotic attractors in three-variable maps with an LCE-spectrum $(+,-,-)$ with $(\Lambda_1 \geq \Lambda_2 \geq \Lambda_3)$. Λ_1 assumed to be ln 2. a) Ordinary chaos $((\lambda_1 + \lambda_2) < 0$ and $(\lambda_1 + \lambda_3) < 0)$, b) Kaplan-Yorke chaos $((\lambda_1 + \lambda_2) > 0$ but $(\lambda_1 + \lambda_3) < 0)$, c) "bi-fractal" chaotic attractor $((\lambda_1 + \lambda_2) > 0$ and $(\lambda_1 + \lambda_2) > 0)$.

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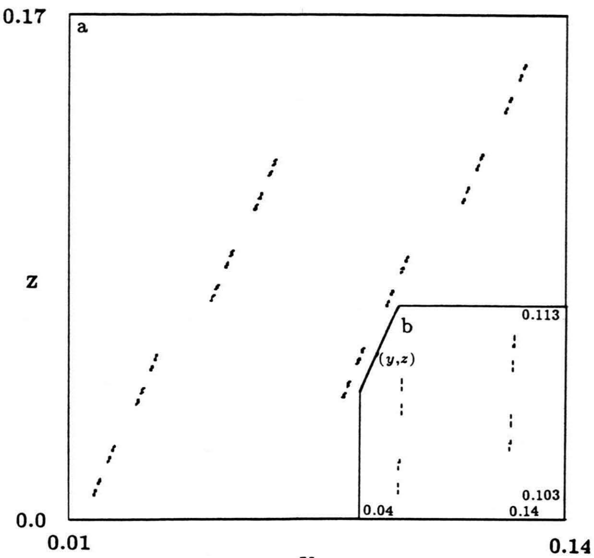

Fig. 2. Dust-like (y, z)-cross-section of an ordinary chaotic attractor of (1) with $A=0.5$, $B=0.0625$, $C=2\cdot10^{-10}$, $\varepsilon=0.1$, (y, z)-plane with $|x-0.25|\leq 0.001$. a) $0.01\leq y\leq 0.14$ and $0.0\leq z\leq 0.17$, b) close up of a) with $0.04\leq y\leq 0.14$ and $0.103\leq z\leq 0.113$.

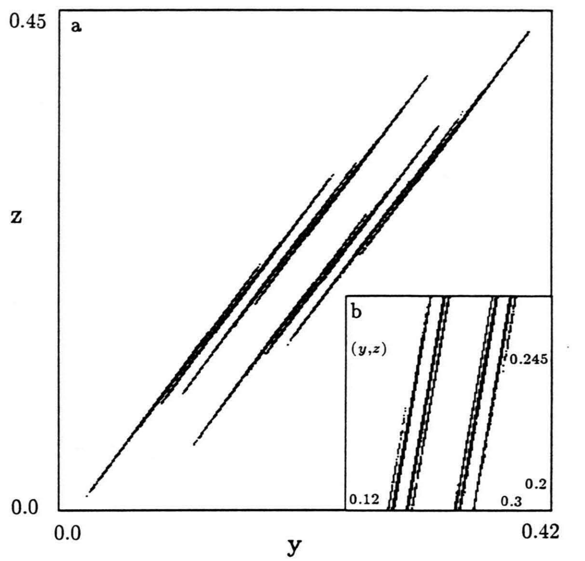

Fig. 3. Striated (y, z)-cross-section of an Kaplan-Yorke chaotic attractor of (1) with $A=1.0$, $B=0.21$, $C=2\cdot10^{-10}$, $\varepsilon=0.1$, (y, z)-plane with $|x-0.25|\leq 0.001$. a) $0.0\leq y\leq 0.42$ and $0.0\leq z\leq 0.45$, b) close up of a) with $0.12\leq y\leq 0.3$ and $0.2\leq z\leq 0.245$.

We will restrict our study to the case of purely *real* eigenvalues (λ_i^r) (the analysis of the complex eigenvalues may be found in [5]). The real eigenvalues restrict the window of stability ($|\lambda_2^r|, |\lambda_3^r| < 1$) to a triangular area in parameter space with $|A| < (B+1)$. The parabola $B = A^2/4$ marks the transition between complex eigenvalues (λ_2^c, λ_3^c) and real eigenvalues (λ_2^r, λ_3^r). The fixed point is a focus above this parabola and a node ($\mathrm{Im}(\lambda_2) = \mathrm{Im}(\lambda_3) = 0$) below.

3. Competition of Local Divergence and Local Convergence

With ($\varepsilon > 0$) the chaos-generating variable x is coupled to the stable fixed point in the (y, z) plane. We find *three* types of simple chaotic attractors with LCEs $\Lambda_i = \ln|\lambda_i|$ and LCE spectrum $(+, -, -)$.

The dynamics of *ordinary chaotic* attractors is characterized with $\lambda_1\cdot\lambda_2^r < 1$ and $\lambda_1\cdot\lambda_3^r < 1$, which means that both rates of convergence (Λ_2, Λ_3) exceed the rate of divergence (Λ_1). The structure of the ordinary chaotic attractors is a striated fractal, a Cantor-set of an infinitely often folded line (*Hénon*-type attractor [6]) giving rise to a fractal dimension $1 < D_f < 2$. The numerically calculated 2-dimensional cross-section (Eq. (1) with $A=0.5$, $B=0.0625$, (y, z)-plane with $|x-0.25|\leq 0.001$, see Fig. 2a, b) simply gives a dust-like Cantor-set of points with dimension $D_{f,\,\mathrm{section}} = 0.5$.

The *Kaplan-Yorke-chaotic* attractors [7] follow from $\lambda_1\cdot\lambda_2^r > 1$ but $\lambda_1\cdot\lambda_3^r < 1$, which means that the rate of divergence (Λ_1) exceeds the rate of convergence with (Λ_2) but is smaller than the rate of convergence with (Λ_3). The fractal structure of the attractors may be visualized as a Cantor-set of smooth sheets which are folded along one direction of state-space ("folded curtain"). According to the Kaplan-Yorke conjecture the fractal dimension rises above the nearest integer value, i.e. $2 < D_f < 3$ in a 3-variable map. The cross-section (Eq. (1) with $A=1.0$, $B=0.21$, (y, z)-plane with $|x-0.25| < 0.001$, see Fig. 3a, b) gives a striated Cantor-set of lines with dimension $D_{f,\,\mathrm{section}} = 1.27$.

In the map (1) a second transition of simple chaotic states is possible with $\lambda_1\cdot\lambda_2^r > 1$ and $\lambda_1\cdot\lambda_3^r > 1$. The *bi-fractal chaotic* attractors [8] are fractalized along two directions of state-space, which means that the rate of divergence (Λ_1) exceeds both rates of convergence (Λ_2, Λ_3). The fractal structure becomes nowhere differentiable with fractal dimension $2 < D_f < 3$. The cross-section (Eq. (1) with $A=1.2$, $B=0.36$, (y, z)-plane

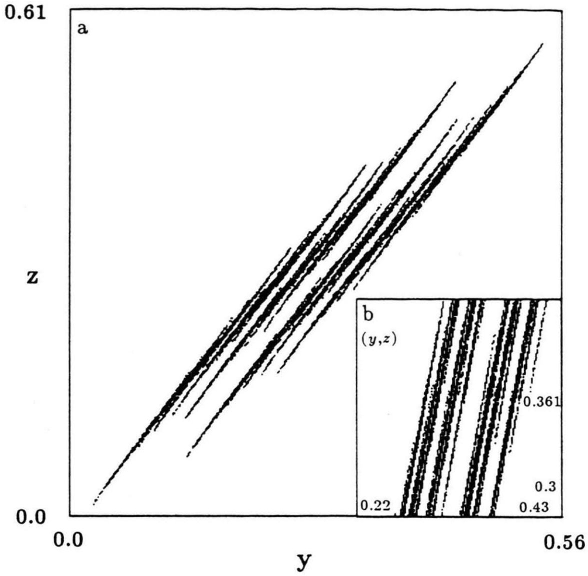

Fig. 4. (y, z)-cross-section of a bi-fractal chaotic attractor of (1) with $A=1.2$, $B=0.36$, $C=2 \cdot 10^{-10}$, $\varepsilon=0.1$, (y, z)-plane with $|x-0.25| \leq 0.001$. a) $0.0 \leq y \leq 0.56$ and $0.0 \leq z \leq 0.61$, b) close up of a) with $0.22 \leq y \leq 0.43$ and $0.3 \leq z \leq 0.361$.

with $|x-0.25| \leq 0.001$, see Fig. 4a, b) reveals a complicated Cantor-set of fractalized line elements with dimension $D_{\mathrm{f, section}} = 1.36$.

4. Discussion

The idea of *dynamical* properties of chaotic attractors (rates of local divergence and local convergence) determining *statical* fractal properties (fractal dimension, smoothness or nowhere differentiability) seems to be universal. The model system of (1) may be seen as prototypical, for we can show [9] that it exhibits dynamical behavior equivalent to a map version of the solenoid [10].

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